Discrete Mathematics 23 (1978) 313-316.


## NOTE

# CHROMATIC UNIQUENESS OF THE GENERALIZED θ-GRAPH

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Received 11 July 1977
Revised 7 February 1978

A generalized $\theta$-graph is a connected graph with 3 paths between a pair of vertices of degree 3. It is shown that any graph having the same chromatic polynomial as a generalized $\theta$-graph, must be isomorphic to the generalized $\theta$-graph.

## 1. Introduction

The *generalized $\theta$-graph*, denoted by $\theta_{d,e,f}$, is a finite, undirected, loopless, simple, connected graph consisting of 3 edge-disjoint paths between 2 vertices of degree 3. All other vertices have degree 2. These paths have lengths $d$, $e$, and $f$ respectively, where $d \leq e \leq f$. The graph has $n = d+e+f-1$ vertices and $d+e+f$ edges. This graph is shown to be *chromatically unique*, i.e., if there exists another graph $Y$ whose chromatic polynomial $\mathcal{P}(Y, \lambda)$ is equal to $\mathcal{P}(\theta_{d,e,f}, \lambda)$, the chromatic polynomial of $\theta_{d,e,f}$, then $Y$ is isomorphic to $\theta_{d,e,f}$.

## 2. Chromatic uniqueness

To establish the chromatic polynomial of $\theta_{d,e,f}$, we can use [2, Theorem 1], which states that the chromatic polynomial of a graph is equal to the sum of the chromatic polynomial of the graph with 1 edge added, plus the chromatic polynomial of the graph with the 2 endpoints (whose edge was just added) identified.

For $d > 1$ we add the edge connecting the 2 vertices of degree 3 to obtain 2 graphs, $G$ and $H$. $G$ consists of 3 cycles with exactly one edge in common, and $H$ consists of 3 cycles with exactly one common vertex. (With $d = 1$, chromatic uniqueness of the $\theta$-graph is shown in Chao and Whitehead [1].)

It is well-known that the chromatic polynomial of the cycle of length $l$ is $\mathcal{P}(C_l, \lambda) = (\lambda-1)[(\lambda-1)^{l-1} + (-1)^l]$. Therefore $\mathcal{P}(\theta_{d,e,f}, \lambda) = \mathcal{P}(G, \lambda) + \mathcal{P}(H, \lambda)$, where

$$\mathcal{P}(G, \lambda) = \frac{\mathcal{P}(C_{d+1}, \lambda) \cdot \mathcal{P}(C_{e+1}, \lambda) \cdot \mathcal{P}(C_{f+1}, \lambda)}{[\lambda(\lambda-1)]^2}$$

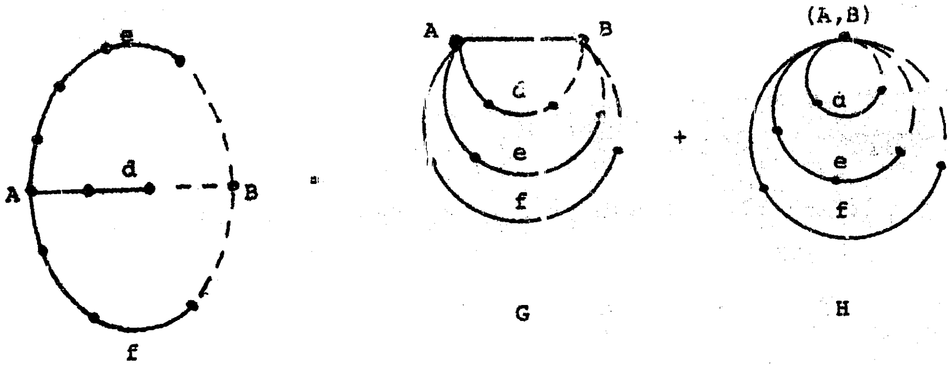


Fig. 1.

and

$$\mathscr{P}(H,\lambda)=\frac{\mathscr{P}(C_d,\lambda)\cdot\mathscr{P}(C_e,\lambda)\cdot\mathscr{P}(C_f,\lambda)}{\lambda^2}$$

by [2, Theorem 3]. It is important to note that $\mathscr{P}(\theta_{d,e,f},\lambda)$ is divisible by $(\lambda-1)$ but not by $(\lambda-1)^2$.

Now let $Y$ be a graph such that $\mathscr{P}(Y,\lambda)=\mathscr{P}(\theta_{d,e,f},\lambda)$. To establish the isomorphism of $Y$ and $\theta_{d,e,f}$, the following properties of $Y$ are necessary and sufficient:

(1) $Y$ is a connected graph with $n+1$ edges. Since $\theta_{d,e,f}$ is connected and has $n+1$ edges, the coefficient of $\lambda^{n-1}$ in $\mathscr{P}(\theta_{d,e,f},\lambda)$ is $n+1$ and $\lambda^1$ has nonzero coefficient. By [2, Theorem 11 and Corollary 14], $\mathscr{P}(Y,\lambda)=\mathscr{P}(\theta_{d,e,f},\lambda)$ means $Y$ has these properties also.

(2) The degree of every vertex of $Y$ is at least 2. Since $Y$ is connected, no vertex has degree 0. If there exists a vertex $v$ of degree 1, then

$$\mathscr{P}(Y,\lambda)=\lambda\mathscr{P}(Y',\lambda)-\mathscr{P}(Y',\lambda)=(\lambda-1)\mathscr{P}(Y',\lambda),$$

where $Y'$ is obtained from $Y$ by deleting the edge joined to $v$, and identifying $v$ with the vertex to which it was connected. For $n\geq 3$, $Y'$ cannot be colored with just 1 color (it is connected and has at least 2 vertices). Thus $(\lambda-1)$ is a factor of $\mathscr{P}(Y',\lambda)$, making $\mathscr{P}(Y,\lambda)$ divisible by $(\lambda-1)^2$. This is a contradiction.

(3) $Y$ must be of the desired form. Since $Y$ has one more edge than vertex and is connected, with no vertices of degree 1, it follows that $Y$ has one of the following three forms:

With form (a), $Y$ consists of 2 cycles, say of lengths $i$ and $j$, connected by an isthmus of length $m\geq 1$. $n=i+j+m-1$ and

$$\mathscr{P}(Y,\lambda)=\frac{[(\lambda-1)\mathscr{P}(C_i,\lambda)]\cdot[(\lambda-1)^m\mathscr{P}(C_j,\lambda)]}{\lambda(\lambda-1)}.$$

[Think of 2 cycles with "tails", where a tail of length $m$ adds a factor of $(\lambda-1)^m$.

The cycles overlap in $K_2$.] In this case, $\mathcal{P}(Y,\lambda)$ is clearly divisible by $(\lambda-1)^2$—contradiction.

Form (b) is similar. Here 2 cycles of lengths $h$ and $k$, $n=h+k-1$, overlap at one point, and

$$\mathcal{P}(Y,\lambda)=\frac{\mathcal{P}(C_h,\lambda)\cdot\mathcal{P}(C_k,\lambda)}{\lambda},$$

again, clearly divisible by $(\lambda-1)^2$, contradicting what was previously noted.

This leaves only form (c)

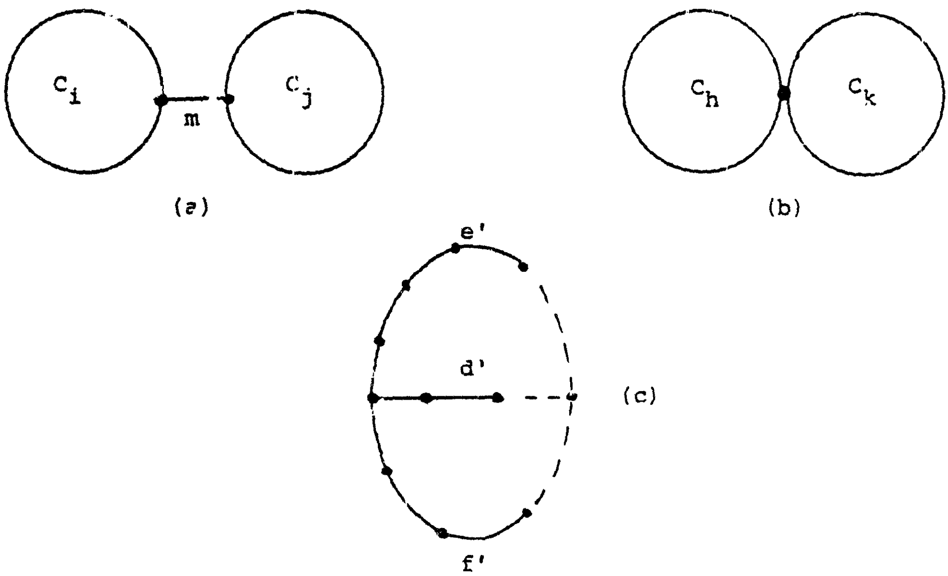


Fig. 2.

(4) If $Y=\theta_{d',e',f'}$ then $d=d'$, $e=e'$, $f=f'$. So far, $Y$ is connected, has $n$ vertices (2 of degree 3, $n-2$ of degree 2), $n+1$ edges, and does not contain an isthmus. Therefore $Y$ must be $\theta_{d',e',f'}$ with $d'\leqslant e'\leqslant f'$. Recall $n=d+e+f-1=d'+e'+f'-1$.

Both $\theta_{d,e,f}$ and $Y=\theta_{d',e',f'}$ contain three cycles. The lengths of these cycles can be ordered as follows:

$$d+e\leqslant d+f\leqslant e+f,$$
$$d'+e'\leqslant d'+f'\leqslant e'+f'.$$

This ordering is a direct consequence of assuming that $d\leqslant e\leqslant f$ and $d'\leqslant e'\leqslant f'$.

*Case* 1. $d+e\neq d'+e'$. Let $m=\min\{d+e,d'+e'\}$. Then the coefficient of $\lambda^{n-m+1}$ in the chromatic polynomials of graphs $\theta_{d,e,f}$ and $\theta_{d',e',f'}$ will be different by Whitney's broken cycle theorem; one graph will contain a broken cycle of length $m-1$ while the other graph does not. This contradicts the fact that $\mathcal{P}(\theta_{d,e,f},\lambda)=\mathcal{P}(\theta_{d',e',f'},\lambda)$.

*Case* 2. $d+e=d'+e'$ and $d+f\neq d'+f'$. Let $m=\min\{d+f, d'+f'\}$. Then the coefficient of $\lambda^{n-m+1}$ in the chromatic polynomials will be different as in Case 1. This is a contradiction.

*Case* 3. $d+e=d'+e'$ and $d+f=d'+f'$. Here, an algebraic argument yields that $d=d'$, $e=e'$ and $f=f'$. Thus $\theta_{d,e,f}$ and $Y=\theta_{d',e',f'}$ are isomorphic.

From the preceding arguments, we have derived the following theorem:

**Theorem 2.1.** *The generalized* $\theta$*-graph is chromatically unique.*

## Acknowledgement

I would like to thank E.G. Whitehead Jr. for his helpful suggestions and encouragement.

## References

[1] C.Y. Chao and E.G. Whitehead Jr., On chromatic equivalence of graphs, in: Y. Alavi and D.R. Lick, eds., Theory and Applications of Graphs, Lecture Notes in Mathematics 642 (Springer, Berlin, 1978).

[2] R.C. Read, An introduction to chromatic polynomials, J. Combinatorial Theory 4 (1968) 52–71.